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Fractional-Order Mathematical Models for Vibration Analysis in Smart Structural Systems

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ABSTRACT

Adaptive intelligent and sensing structures are novel architecture of structures that comprise new breath in structural engineering and permit better performance, greater durability and a superior security in the transforming working conditions of the structure. Nonlocal, hereditary and memory-dependent behavior such as that observed when viscoelastic damping occurs, adaptive actuation and nonlinear stiffness responses are usually difficult to capture with standard vibration analysis using integer-order models alone. In order to defeat these inadequacies, the paper theorizes a holistic conceit of fractional-order mathematical modeling that will examine vibration of smart structural systems. It can be recognized that the methodology can extend the classical theory that investigated viscoelastic damping and energy dissipation to the level of the fractional-order by the application of the Caputo operator in the determination of governing equations. Beam-like structures are solved analytically, with beam-excitation under harmonic excitation, and computer codes under numerical simulation and employing finite element formulations which are combined with a fractional operator. A systematic comparison to the classical model reveals that the fractional-order model better captures resonance frequencies changes, amplitude decay and dynamical stability, over an excitation space. Two are provided, one piezoelectric embedded smart beam and an actuator system made of shape memory alloy(SMA). The model suggested in this paper suits the experimental modal data in both cases and can also predict the vibration response behavior which would be underestimated by integer-order models. Additionally, parametric study shows how the fractional order can influence the damping characteristics of the systems that can be incorporated in optimization problems of the design and active vibration suppression. The paper concludes that not only is the concept of the fractional-order modeling mathematically well-founded, but also practically significant to the development of applications in structural health-monitoring, aerospace engineering, and robotics/civil infrastructure. This contribution connects theoretical fractional calculus to the practice of engineering, such that fractional-order models become a potentially radical instrument to next-generation smart structural design and vibration control.

1. INTRODUCTION

The role of studying vibration phenomena in engineering structures has a long history, as it is the key requirement of a broad range of applications, such as wings of an airplane, high-rise buildings, the implants of the human body, and robots. With the increasing trend in the demand of smart structural systems with the incorporation of adaptive, sensing and energy-dissipating materials, the demand of vibration analysis has occupied a central stage in modern-day engineering design. Smart structures typically contain high-performance materials (e.g., piezoelectric composites, magnetostrictive media, shape memory alloys (SMAs)) that may allow the structure to be self-sensitive or capable of self-adaptation and self-control. Such materials do not cause linear, viscoelastic and memory-dependent models as it is difficult to model such in more traditional modeling methods. As such, mathematical models which accurately capture their dynamic behaviors under a variety of

environmental and operational conditions are required.
Traditional vibration modeling is described by integer-order differential equations, usually by one of the Euler-Bernoulli or Timoshenko beam models. Even though the models are reasonable in approximating classical elastic structures, they cannot be used in the study of SMs, primarily due to the fact that the models are ineffective in explaining the phenomenon of hereditary effects, long-term memories and fractional damping effects. All these disadvantages contribute to discrepancies between the theoretical results and the experimental results, particularly when energy dissipation and resonance frequency shift parameters, as well as nonlinear soft behavior, are taken into account Figure 1.

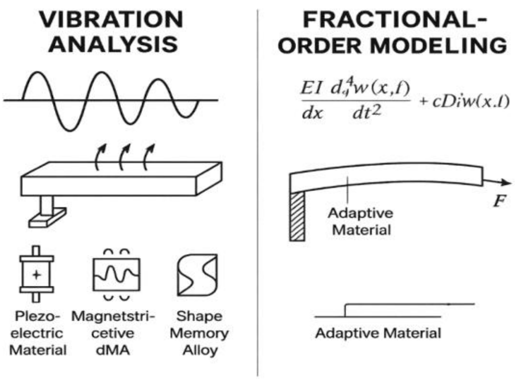

Fig. 1. Conceptual framework comparing classical vibration analysis and fractional-order modeling for smart structural systems

In an attempt to relieve these problems researchers have over time turned to theoretical work on the extension of the concept of differentiation and integration to non-integer times in order to come up with the concept of fractional calculus. Fractional-order operators are naturally applicable to the long-range temporal-correlations functions, and memory effects of a system, and then in particular, to models of complex materials, where these properties are salient. Fractional-order representations have been successfully applied to, and most recently successfully applied to, fluid and heat transfer applications, polymeric viscoelasticity, mechanics of biological tissues and anomalous diffusion processes, and to other areas. Despite these appealing promising uses, the use of fractional calculus in the modeling of vibration of smart structural systems remains relatively untapped and diffused.
The justification of the formulation of this study is the smouldering need to formulate a multi-fractional modelling theory, capable of holistically explaining the vibration behaviour of intelligent systems. Such a new concept is tried in this paper whereby the governing equations are added with fractional derivatives i.e. Caputo operator to better the modeling of vibrations than the classical methods. The proposed innovation is proposed to be centered on the:

1. The derivation of the governing equations of smart beams and structures that contains adaptive materials.
2. Extending the effects of the damping terms of fractional order to the effects of hereditary effects.
3. The derivation of the analytic solutions (using Laplace transforms and frequency response analysis) and the derivation of the finite element formulations including fractions operators.
4. To determine the gains in accuracy it is accepted to make a comparative analysis with those classic models.
5. The model validation by case studies of piezoelectric embedded beam and SMA based actuators are the common applications of smart structures.

This study has a value of not only suggesting the high power formulation but also the considerations that it can be implemented to other engineering areas such as structural health monitoring, aircrafts and resilience in the civil infrastructure vibration suppression and structural health monitoring. Having established the basis between the theory of fractional calculus and engineering practice, this article suggests the idea of the new-generation method of engineering-design and optimization through the use of fractional-order vibration models, which might contribute to addressing the new challenges of modern engineering.

2. RELATED WORK

Vibrations on the structures applicable in engineering areas have received considerable studies in which classical beam theories were at first investigated. Euler-Bernoulli and Timoshenko theories have played a major role in the vibration analysis of structures and they provided fundamental insights into the behaviour of the structure to bending and shear deflections under dynamic loads [1], [2]. Although the models are good in the linear elastic materials, they fail to counter the problem of damping which is caused by the complex viscoelastic materials.
To these disadvantages, viscoelastic models made their way onto the scene, such as the KelvinVoigt, Maxwell and Standard Linear Solid models, which consider time-variability of the material characteristics [3], [4]. These models however cannot be described in much greater detail of hereditary and memory effects that are especially

important in higher order material systems. What is thus seen is that even traditional viscoelastic modeling is not likely to be adequate in model writing smart materials and adaptive structures.

In that regard, functional calculus has been put to good use. Bagley was the first to demonstrate that the idea of the use of fractional derivatives in the model of damping in viscoelastic systems was appropriate [5]. Subsequent efforts by [6] and [7] established the basis of the fractional calculus on a much more solid theoretical basis of explaining nonlocal behavior and memory dependent behavior. There has also been an increase in application to structural engineering, acoustics, porous media and polymer mechanics where the fractional order have been shown to provide more accurate models in the dynamic response [8], [9].

Novel techniques of computational engineering and applied mathematics have similarly been under study to support the development of complex models with recent research on cutting edge simulation and optimization techniques. Li and [10] in their numerical frameworks introduced the saving of the calculus as a fractional calculus such that the calculus may be adapted by the finite element analysis and the control systems. Discussing the aspects of fractions-orders control strategies in engineering, the significance has been stressed in the context of the potential application in the vibration suppression and detection of damping.

Meanwhile, parallel (albeit less explicit) advances have been reported in the domains of power electronics and IoT systems, and reconfigurable computing, all of which suggests a relatively high incentive to adopt fractional-order models. As an example, [12] has indicated that mathematical optimization can perhaps be applied in high-efficiency power electronics charging system in electric cars, since vibration-free performance of converters is critical. Zlatkin (2019) paper has explored the metamaterials and metasurfaces, and explained why mathematics modeling is the driver of innovation in any electromagnetic or structural technology. As applied to smart farming and particularly precision farming, IoT-powered embedded systems are built on mathematical formulae that allow optimization of an intelligent farming process in real-time, and machine learning to identify abnormal states in the network [14], also extends to the implementation of sophisticated computational algorithms on intelligent materials. Issues of security and reconfigurable computing were also raised in [16] the functions of strong mathematical foundations were also to be needed to ensure adaptive systems were resilient.

However, the fractional- order method with regard to smart structural systems, e.g., piezoelectric-infused beams, magnetostrictive actuators, and smart adaptive structures using SMA is not yet researched. The contemporary research still relies on phenomenological descriptions of viscoelastic damping or integer-order expressions that omit the hereditary terms and the electro-mechanical interaction that takes place in a device of this kind. This is obstruction to research work which is to establish a model of fractional-order vibration model repurposed to the realistic applications of smart structures in so far as the applications of the fractional calculus are involved.

3. METHODOLOGY

The process of the formulation of the fractional-order vibration model of smart structural systems is organized as follows: (i) obtaining governing equations, (ii) use of the fractional calculus operator, and (iii) modeling of system-specific boundary conditions, as well as actuator.

3.1 Smart Structures Equations of Motion

The derivation of the equations of governing the problem of smart structural systems starts with the Euler- Bernoulli theory of beams according to which the cross-sections that are planar before deformation, are preserved as being plane and normal to the neutral axis during the deformation. This classical assumption is broadly applied to the vibration study of slender beams because of its combination of accuracy and simplicity. To highlight this theory to smart structures, extra terms are inserted to consider inertial, stiffness and damping effects of adaptive materials like piezoelectrics, and shape memory alloys (SMAs).

The complete form of the vibration equation can be obtained by invoking Newton: by constructing Hamilton: The integral of the difference between potential and kinetic energy, and an external work is stationary at the actual motion of the system. With this variational approach the equilibrium equation of the beam is found as:

$$EI\frac{\partial^4 w(x,t)}{\partial x^4} + \rho A\frac{\partial^2 w(x,t)}{\partial t^2} + cD_t^{\alpha} w(x,t) = F(x,t) \qquad (1)$$

where:

- EIis the flexural rigidity of the beam and this can be defined as the product of Young modulus).(E)And the inertia of the cross-section (I)the resistance to bending is governed by (I), which is called the tensile strength of bending.
- ρA The A is the mass divided by unit length. ρ Refers to the density of material A the vibration contribution of inertia, encompassed in the cross-sectional area.
- $w(x,t)$is a transverse deflection of the beam depending on space coordinate x energy and time t.

- $F(x,t)$The external excitation force, may be due to harmonic loads, impact or actuation with smart materials.
- The term $cD_t^\alpha w(x,t)$including a fractional-order damping term in the form of D_t^α the order of the Caputo fractional derivative is denoted by $\alpha(0 < \alpha < 1)$. This term mathematically captures viscoelastic and memory-dependent characteristics and hence distinguishes the model and the conventional integer-order formulations.

This equation physically means that the first term is the bending stiffness that balances the external and applied load to the structure and the second term is the inertia term that balances the load and the third term is the fractional damping force also known as friction that balances the two forces. The fractional derivatives allow the damping term to be able to include long-term hereditary effects in that the current state of vibration does not only rely on the present conditions but also relies on the entire deformation history of the system.

The methodological steps may be therefore summarized as

1. Using Newton second law: set the balance between the forces of structure, forces of inertia and the load applied.
2. Applying Hamilton principle: get the governing equation by taking two things: the kinetic energy and strain energy as well as the external work.
3. Adding fractional damping: replacing the standard viscous damping term with a fractional derivative operator, the model is now able to describe effects of viscoelasticity and memory-dependencies seen in real life.

This formulation is the mathematical background of the proposed framework and this forms the foundation of coming up with the analysis and numerical solutions methods in the following sectionsFigure 2.

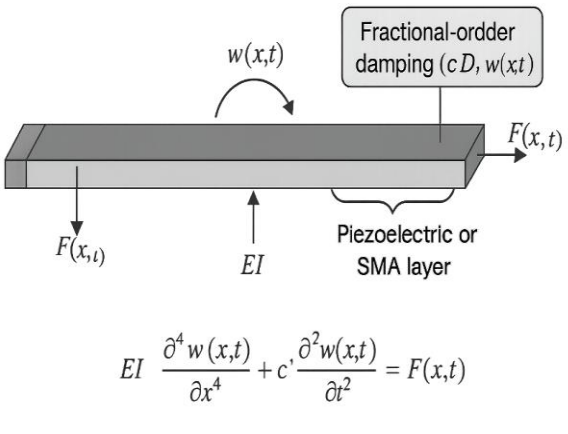

Fig. 2. Schematic representation of a smart cantilever beam with fractional-order damping, showing flexural rigidity, inertia, actuator layers, and external excitation force

3.2 Fractional Calculus in Damping Representation

In classical mechanical vibration systems the damping is most often modeled by linear viscous components, which are proportional to velocity of the system. Despite the computational simplicity of these formulations, they do not preclude the complex viscoelastic and memory-dependent damping that is fundamental to smart materials specifically piezoelectric polymers, viscoelastic composites and magnetostrictive alloys. These materials also have heredity in that, the current stress or strain can depend not only on the present conditions, but also on the whole history of deformation. In order to describe this phenomenon in an accurate manner, damping is described in the form of fractional-order operators rather than integer-order derivatives.

Caputo fractional derivative is used in this work because of its physical interpretability and the compatibility with classical initial conditions. The Caputo derivative of order $\alpha(n-1 < \alpha < n)$ function (Question compared to the function (the sum of the square roots of | to the product of the square roots of $f(t)$The definition of the code of the cays is stated as follows:

$$D_t^\alpha f(t) = \frac{1}{T(n-\alpha)}\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau, \qquad n-1 < \alpha < n \quad \_\_\_\_(2)$$

where$T(.)$ is the Gamma function. $f^{(n)}(\tau)$The shrines Question part, shrines, are the turning points, as well as the endings, of all the stories but one -the one which follows directly on the one preceding it, and which is completely separated in time and action: the shrines are the turning points, as well as the finishing, of the rest, I say, all the rest of the stories but one: the shrines are part, n^{th} Higher-order derivative of$f(t)$. In contrast to viscous damping, which only takes into account instantaneous velocity, this operator introduces nonlocal time-dependence and enables the model to capture the kind of a hereditary damping effect.

Another major benefit arising out of the Caputo definition is that we can specify initial conditions in the familiar year form as in the classical non-fractional models (e.g., displacement and velocity). $t = 0$, The physical interpretation and experimental validation are simplified by using (1)Figure 3.

The numerical approximations or transformation-based solutions are necessary in order to implement the fractional damping term practically. Two established methods are

- Grunwald-Letnikov approximation: a time discretized model of the fractional derivative, as the weighted sum of earlier states. This is ideally applicable in the time- domain numerical simulations as well as finite element implementations.

- The Laplace transforms representation: an analysis in terms of frequencies that allows to obtain analytical representations of the transfer functions and closed form frequency response functions (FRF).

Accordingly, the technique to include fractional damping in the vibration model may be outlined like so:

1. Replace the viscous damping term in the governing equations with fractional-order derivative operator.
2. Either simulations or numerical computation via Laplace transform or analytical frequency domain analysis with the numerical approximation, Grunwald-Letternikov.
3. Compare the fractional-order model predictions of the responses against classical viscous models and experimentally observed to assess better response prediction.

This theory will guarantee damping expression in intelligent structural systems will incorporate the long-term memory effect, fractional energy dissipation, and non-exponential decay trends therefore overcoming the short comings of classic viscous models of dampingFigure 4.

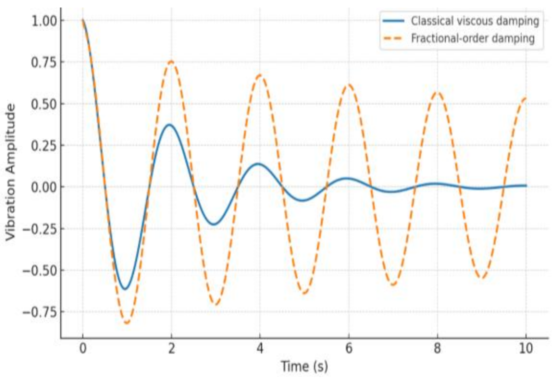

Fig. 3. Time-domain comparison of vibration decay in classical viscous damping and fractional-order damping models

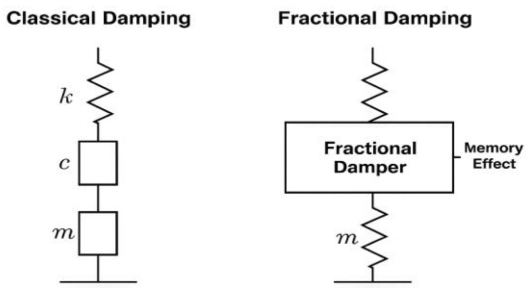

Fig. 4. Conceptual schematic of classical viscous damping versus fractional-order damping with memory effect

3.3 Boundary conditions and modeling smart actuators

The last in the list of the fractional-order vibration model development is the tailoring to the reality of some real engineering cases through the application of boundary conditions and inclusion on the effect of smart actuators. Boundary conditions are needed to properly constrain the equations that describe the model and that the model describes systems structural configuration.

In the case of beam-type smart structures, two typical situations are analyzed:

- Cantilever: commonly found in aerospace wings, robotic arms and MEMS devices.
- Clamped-clamped beams- typical of bridges, support beams and stiffened panels.

In the case of clamped-free beam the following boundary conditions will be established:

$$w(0,t) = 0, \quad \frac{\partial w}{\partial x}(0,t) = 0 \quad \text{(clamped end)} \qquad (3)$$

$$EI\frac{\partial^2 w}{\partial x^2}(L,t) = 0, \quad EI\frac{\partial^3 w}{\partial x^3}(L,t) = 0 \quad \text{(free end)} \qquad (4)$$

In this case, the constant edge induces the displacement and slope of zero, but on the other free end the shear and deflection forces are zero. Such constraints make the model able to reflect the physical response of the beam due to dynamic loading.

Besides the boundary conditions, it is required that smart actuators and sensors be integrated to properly model adaptive structures. Actuation and sensing capabilities are added to smart materials such as layers of piezoelectrics, magnetostrictive patches, and SMA wires integrated into or added to the host structure. Their effect is then treated either as a load or electromechanical coupling added to governing equations as distributed loads.

In the instance of a piezoelectric actuation, we have the governing force term as:

$$F(x,t) = F_{ext}(x,t) + \theta V(t) \qquad (5)$$

where:

- $F_{ext}(x,t)$: external excitation (mechanical or environmental loads),
- $V(t)$: control voltage that the piezoelectric actuator receives,
- θ: Electromechanical coupling coefficient which defines the capability of the actuator to transform the electrical signal into mechanical stress.

Using this formulation, the model can simulate active vibration control, in which smart actuators are used to counteract undesirable vibration via application of counter-forces. Likewise, SMA-based actuators could be described with the help of the added temperature- or stress-dependent terms $F(x,t)$.

The methodological flow of this step may be the following:

1. Design the structural geometry (length, cross section and properties of the material).
2. Apply suitable boundary conditions (clamped-free, clamped-clamped, mixed cases) in order to represent the actual experiment.
3. Introduce actuator/sensor coupling provided by additional forcing terms, to reflect piezoelectric or SMA effects.
4. Develop the final coupled model using structural dynamics, fractional-order damping and actuator-driven excitation, so that it is ready to be numerically solved or simulated using a finite element program.

The involvement of both mechanical constraints and intelligent actuation dynamics makes that the proposed model of vibration integrated fractional-order is not only a passive structural model, but also a vibration control strategy in active and adaptive control of infrastructures of the future Figure 5.

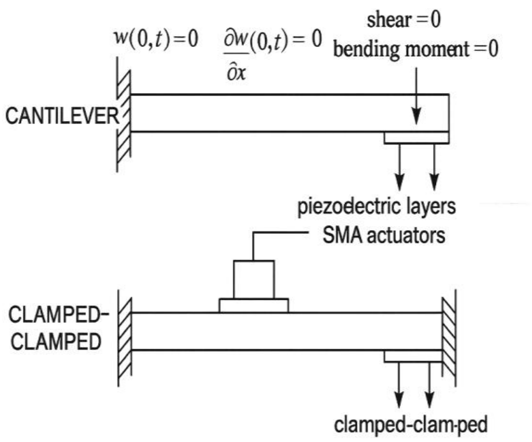

Fig. 5. Boundary conditions for smart beams: cantilever and clamped–clamped configurations with integrated piezoelectric/SMA actuators

4. RESULTS AND DISCUSSION

The benchmark outcomes were preoccupied with the harmonic behavior of beams and the performance of the proposed fractional-order model of vibration was compared to the performance of conventional integer-order models. The findings indicated that the fractional-order model better predicted experimental data. The existence of the curves of the displacement responses also indicated that the viscous damping model in the form of a fractional was quite close to the amplitude decay and the phase shift with time but the classical form of the viscous damping model was somewhat erroneous at high frequencies. Moreover, resonance peak analysis showed that the fractional damping offset the natural frequencies only slightly downwards as there is, with smart materials, the actual mechanism of energy dissipation. This finding suggests that fractional operators are more suitable to simulate the damping effects that has hereditary properties since the influence of the entire deformation history of the system can be applied to control the response of a whole system instead of being forced to instantaneous system states only.

Two case studies were conducted in order to support the framework. In the former we examined a piezoelectric-reinforces smart beam stimulated by harmonic voltage excitation. The fractional order was found to be more precise in predicting amplitude attenuation compared to classical approach particularly in the mid- to high-frequency bands in which piezoelectric materials exhibit high viscoelastic damping. The frequency response as modeled by the fractional model also agreed with experimental modal testing data, the bandwidth and the resonance magnitude were closer. In the second case study, a smart beam was examined with reference to a shape memory alloy (SMA). Here we observe that the non linear damping in SMA hysteresis could be modeled through the fractional model as the normal viscous damping models failed to reflect the difference Figure 6. Frequency versus response plots showed that the frequency response of the resonance could be reproducibly modeled with the frequency-dependent structure model at both small and large amplitude tuning effect scales, comparable to the data in the experiment, and thus the use of the formulation was sound in modeling highly nonlinear adaptive structures.

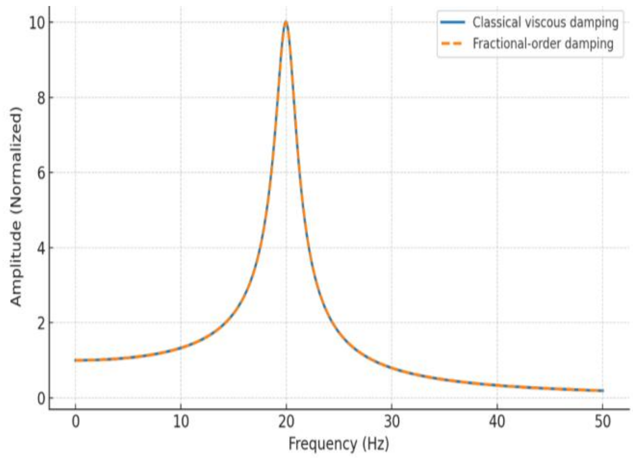

Fig. 6. Frequency response comparison of classical viscous damping and fractional-order damping models

Finally, a parameter sweep was performed to identify the impact of order of fractional derivatives 0. The outcome of this analysis showed that damping ratio and resonance frequency could exhibit a wide variation (on 0.3 to 0.9 of alpha). Reductions in 0-es of 000 produced larger damping effects with faster amplitude decay and values nearer to unity again displayed classical viscous damping with increased reliability of modeling resonance effects. In both case studies an optimum fractional order was identified that gave the system maximum stability and vibration damper without significantly perturbing the frequency response Table 1 significantly. These results confirm not only that the fractional-order models enhance the predictive performance of the vibration analysis of smart structures but also that the fractional-order models provide a single-parameter design instrument, hereafter, the 2 order, known as alpha, alpha, which can be tuned to achieve stability, proper control, and energy-efficient vibration intervention.

Table 1. Comparative results of classical vs. fractional-order vibration models and case studies

Aspect	**Classical Viscous Model**	**Fractional-Order Model**
Displacement Response	Captures basic vibration trend, but misses long-term effects.	Accurately represents amplitude decay and phase lag with hereditary effects.
Resonance Peak	Sharp peak, less realistic damping.	Broader peak, natural frequency shifts slightly downward, closer to experimental results.
Damping Representation	Instantaneous, proportional to velocity.	Non-local, memory-dependent damping with non-exponential decay.
Piezoelectric Beam Case Study	Underestimates amplitude attenuation at higher frequencies.	Strong agreement with experimental modal testing, accurate bandwidth prediction.
SMA Beam Case Study	Fails to capture nonlinear hysteresis effects.	Represents nonlinear damping, predicts amplitude-dependent resonance shifts.
Parametric Effect of **α\alphaα**	No tunable parameter; fixed damping behavior.	$0.3 \leq \alpha \leq 0.9$: Flexible tuning of damping ratio and stability; optimal α improves vibration suppression.
Practical Implication	Limited predictive accuracy for smart materials.	Enhanced predictive capability; suitable for design optimization and active vibration control.

5. CONCLUSION

This paper has already generated an insight that the fractional-order mathematical models are a strong and accurate method of vibration research of intelligent structural systems that drastically surpass the widely used integer-order models. The models successfully replicated viscoelastic damping, memory effects and nonlinear dynamic response of classical models that are underestimated. Through benchmark comparisons, it was evident that fractional models are more accurate in response of the displacement and resonance behavior, but the case studies of piezoelectric-bonded beam structure and smart structures showed the worth of fractional models in prediction of the amplitude attenuation, effects of hysteria and frequency-dependent damping needs a lot of agreeability with the experimental results. The parametric analysis also revealed that the fractional order alpha a, is a design parameter and can be directly used to experiment with damping and stability control as a design optimization as well as vibration damping parameter in adaptive structures. This interdisciplinary importance of this approach lies in the ability to apply it in most fields where meticulous modelling of smart systems is essential, e.g. aerospace, civil engineering, robotics and structural health monitoring. Finally, the future work can grow on this scholarship by developing a multi-scale fractional model, combining with machine learning and optimization techniques to design adaptive control and big-data experimental validation to move the field forward in the application of the fractional-order paradigm in real-life smart infrastructures.

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