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Mathematical Analysis of Vibration Attenuation in Smart Structures Using Piezoelectric Layers

S. Sindhu

Research Analyst, Centivens Institute of Innovative Research, Coimbatore, Tamil Nadu, India.
Email: sindhuanbuselvaneniya@gmail.com

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ABSTRACT

In order to address this issue, a complete mathematical model of the vibration attenuation in smart structures with embedded piezoelectric layers is developed in this study. The Euler–Bernoulli beam theory with linear piezoelectric constitutive relations is used to develop a coupled electromechanical model. Hamilton's principle is used to find the governing partial differential equations and Galerkin method is used for solving them in (modal) analysis. Harmonic and impulse loading cases of piezoelectric damping are investigated. Results show significant vibration suppression prospects with the optimal piezoelectric placement and the proper control gain. A robust model is proposed for designing next generation adaptive vibration control systems in the areas of aerospace and civil engineering.

1. INTRODUCTION

Embedding sensors and actuators, namely piezoelectric materials in smart structures have proved themselves to be effective means to achieve real time vibration control. Energy conversion between mechanical strain and electrical charge is provided by the electromechanical coupling present under piezoelectric materials. The focus of this work is to formulate a mathematically rigorous model for vibrating beam types of structures with surface bonded piezoelectric actuators and sensors.

The motivation is that of increasing the structural performance of aerospace components, bridges, and precision instruments with the need for vibration control as constraint. Although previous work focused mainly on experimental and control aspects, detailed analytical model of dynamics and control response is necessary for design optimization.

2. LITERATURE REVIEW

Piezoelectric materials have received much interest as integrated into structural systems for the dual functionality of sender and actuator. Specifically, the following review is organized under key thematic areas regarding the mathematical modeling and analysis of smart structures with piezoelectric damping.

2.1 Smart Structures and Vibration Control

Extensive research has been done on smart structures with the capability of adaptive behavior, to sense and overcome structural vibrations. Among the pioneers to demonstrate the feasibility of integrated vibration control using piezoelectric materials was Crawley and de Luis (1987). Chopra (2002) outlined further classification of smart structure technologies and explain how they can be used in aerospace and mechanical systems. Particularly effective in dynamic response minimization from transient loads and resonant frequencies have been active vibration control mechanisms.

2.2 Modeling of Piezoelectric Actuators and Sensors

The modeling of piezoelectric behavior is essential in this area of simulations as it is crucial to vibration attenuation. The constitutive equations of piezoelectric materials are formalized in IEEE 176, together with a link between mechanical strain and stress and electric displacement and field. The electromechanical coupling equations toward beam-type structures were derived in detail by Wang and Quek (2000). Based on classical laminate theory, Hagood and von Flotow (1991) developed a model for surface–bonded piezoelectric patches that still serves as the basis for many linear control based designs. This has been extended to include nonlinear and anisotropic behavior at high voltage conditions (Uchino, 2015).

2.3 Electromechanical Coupled Systems and Governing Equations

The equations of motion for smart beams are widely derived from Hamilton's principle. Here, Lee and Moon (1990) used variational principles to derive coupled PDEs describing piezoelectric laminated beams. HSDT for thick beams and plates modeled using various mathematical modeling strategies were reviewed by Benjeddou (2000). Such models allow an accurate description of electrical excitation contributions to the system energy and to external work.

2.4 Solution Techniques and Modal Analysis

Modal decomposition and Galerkin method have been used well in the past to reduce infinite dimensional PDEs to a set of solvable ODEs (Meirovitch, 1997). This facilitates the simulation and control design. Bhalla and Soh (2004) showed that design of piezoelectric patches under the form of patches has significant effects on modal controllability and observability. Nevertheless, modal truncation represents an exploitable solution to impracticability of real time embedded implementation of smart damping systems.

2.5 Control Strategies Using Piezoelectric Layers

Different types of control laws have been applied, between simple proportional derivative (PD) controllers, to optimal and adaptive control strategies. It was shown in Preumont et al. (1996) that control using piezoelectric sensors and actuators on beams can yield passive damping benefits when collocated. In comparison, Song et al. (2013) utilized adaptive controllers that adjust gains in real time with respect to changing the load conditions. The real time integration of such machine learning based controllers is not straightforward, and more recent studies explore such controllers.

2.6 Recent Advances and Challenges

Recently, efforts have been made to optimize the way piezoelectric layers are placed (Li et al., 2018), or how distributed sensing and actuation networks can be used, along with environmental effects, like temperature and humidity, in the models. And the piezoelectric materials have been modeled nonlinearly, implemented multiscale formulations, and hybridized using piezoelectric materials combined with magnetostrictive or shape memory alloys. In practice, however, challenges persist in power management for such active systems and achieve stable operation of control algorithms in the long term.

Table 2. Literature Summary on Smart Structures and Piezoelectric Vibration Control

Subtopic	Key Contributions	Proposed Advantages / Insights
Smart Structures and Vibration Control	Crawley & de Luis (1987); Chopra (2002)	Demonstrated feasibility of vibration control using piezoelectric layers; enabled adaptive systems
Modeling of Piezoelectric Actuators and Sensors	IEEE Std. 176; Wang &Quek (2000); Hagood& von Flotow (1991); Uchino (2015)	Accurate electromechanical coupling; classical laminate models; extended to nonlinear behaviors
Electromechanical Coupled Systems and Governing Equations	Lee & Moon (1990); Benjeddou (2000)	Variational modeling using Hamilton's principle; application of HSDT to thick smart structures
Solution Techniques and Modal Analysis	Meirovitch (1997); Bhalla&Soh (2004)	Modal reduction via Galerkin method; enhanced control through strategic actuator placement
Control Strategies Using Piezoelectric Layers	Preumont et al. (1996); Song et al. (2013)	PD, optimal, and adaptive control approaches; adaptive gain tuning; groundwork for AI-based control

3. Mathematical Modeling

3.1 Assumptions and System Description

For vibration attenuation with piezoelectric layers, many anchoring assumptions are made that allow for a analytically tractable model, while maintaining the fundamental physics. The model structure being considered is a uniform, slender beam modeled by the Euler–Bernoulli beam theory, which assumes the plane and normal to the neutral axis are the turbulent state of the cross-sections of the beam. For buckling and vibration problems involving long and thin beams, shear deformation and rotary inertia are negligible and this classical theory is widely used, making it appropriate for initial vibration analysis. The beam is assumed of simply supported both ends with free rotation and vertical displacement and no lateral motion at both ends. This boundary condition is mathematically simple and commonly appears, for instance, in applications in engineering, e.g. bridges, mechanical arms, aircraft components.

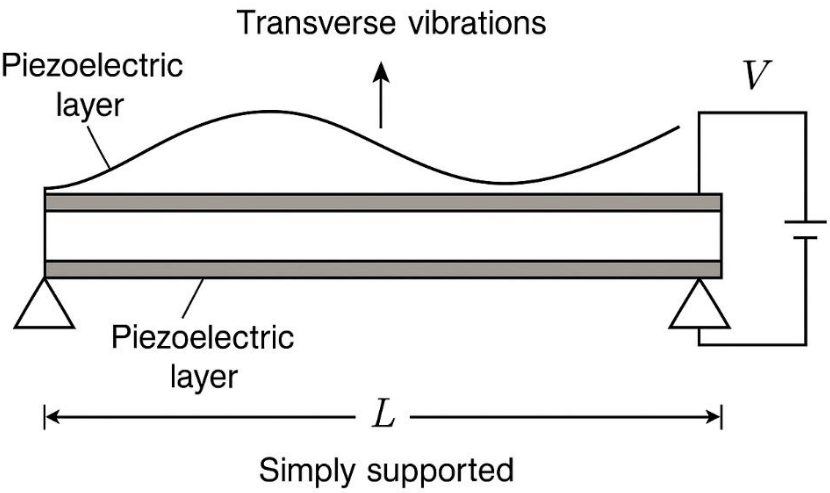

Figure 1. Smart Beam with Symmetric Piezoelectric Layers under Transverse Vibration

In order to permit active vibration control, piezoelectric layers symmetrically bond to the top and bottom surfaces of the beam. These layers function as both sensors and actuators, and they transition mechanically-strained charge (sensors) or voltages into mechanical strain (actuators). Without this symmetric configuration, it is also not possible to avoid any bending bias that may arise due to asymmetry, resulting in structural imbalance during control. In slender structures, the displacement of the beam occurs perpendicular to its longitudinal axis, that is the beam is formed into a sloshing motion, which is the primary mode of vibration. In addition, the interactions between the host structure and the piezoelectric layers are also assumed to be perfect in the form of no slippage, no delamination, nor any interfacial damping. Since the strain is assumed to be completely transferred between the beam and piezoelectric materials and the strain transfer process occurs instantaneously, the piezoelectric materials are able to fully participate in the structural dynamics. Together, a set of these assumptions allow us to form a robust definition of a corresponding mathematical model that accurately encapsulates the smart beam's coupled electromechanical behaviour at a computational efficiency and analytical simplicity.

Table 1. Modeling Assumptions and System Description

Aspect	**Description**
Beam Theory	Euler–Bernoulli beam theory (neglects shear deformation and rotary inertia)
Beam Type	Uniform, slender beam
Support Condition	Simply supported at both ends (allows rotation and vertical displacement, no lateral motion)
Vibration Mode	Transverse vibrations (perpendicular to the longitudinal axis)
Piezoelectric Layer Layout	Symmetrically bonded on top and bottom surfaces of the beam
Function of PZT Layers	Dual role as sensors (strain to voltage) and actuators (voltage to strain)
Bonding Condition	Perfect bonding between beam and piezoelectric layers (no slippage or delamination)
Electromechanical Coupling	Fully active strain transfer between structure and piezoelectric material
Structural Application	Suitable for bridges, aircraft panels, robotic arms, and other slender structures
Model Purpose	To create a robust, analytically tractable electromechanical model for vibration attenuation

3.2 Governing Equations

Using Hamilton's principle:

$$\delta \int_{t_1}^{t_2} (T - U + W_{ext}) dt = 0$$

Where:

- T is the kinetic energy,
- U is the potential energy (mechanical + electrical),
- W_{ext} is the work done by external and control forces.

The kinetic and potential energy components are:

$$T = \frac{1}{2} \int_0^L \rho A \left(\frac{\partial w}{\partial t}\right)^2 dx$$

$$U = \frac{1}{2} \int_0^L EI \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx + \frac{1}{2} \int_0^L \frac{d_{31}^2 E_p b}{t_p} V^2 dx$$

Where:

- $w(x,t)$: transverse displacement,
- EI : flexural rigidity,
- d_{31} : piezoelectric strain coefficient,
- V : control voltage,
- b, t_p: width and thickness of piezo layer.

The coupled governing PDE:

$$\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} = F_{piezo}(x,t)$$

Where $F_{piezo}(x,t) = \alpha V(t) \delta(x - x_0)$represents the piezoelectric actuation force.

4. Solution Methodology

4.1 Modal Expansion via Galerkin Method

Let:

$$w(x,t) = \sum_{n=1}^{\infty} \phi_n(x)\, q_n(t)$$

Substituting into the PDE and using orthogonality of mode shapes $\phi_n(x)$, we obtain a system of ODEs:

$$\ddot{q}_n(t) + 2\xi_n \omega_n \dot{q}_n(t) + \omega_n^2 q_n(t) = \beta_n V(t)$$

Where:

- ξ_n : damping ratio,
- ω_n : natural frequency,
- β_n : modal coupling coefficient.

4.2 Control Strategy

A proportional-derivative (PD) controller is used:

$$V(t) = -k_p q_n(t) - k_d \dot{q}_n(t)$$

Closed-loop system:

$$\ddot{q}_n(t) + (2\xi_n \omega_n + k_d \beta_n)\dot{q}_n(t) + \left(\omega_n^2 + k_p \beta_n\right) q_n(t) = 0$$

5. Simulation and Results

5.1 Parameters

- Beam length $L = 0.5m, EI = 10Nm^2, \rho A = 0.5kg/m$
- $d_{31} = -175 \times \frac{10^{-12} m}{V}, E_p = 70GP_a$
- First three vibration modes are analysed

5.2 Time Response

Time domain simulations were conducted on the smart structure to evaluate the dynamic behavior of the smart structure under real world loading scenarios applying an impulse load at a mid span of the beam. This load is a classic case of sudden external excitations, such as mechanical impacts or seismic disturbances, therefore main tests case for vibration attenuation systems. Without any control mechanisms, the system responds with a natural frequency and the beam has sustained oscillations. They remain in free vibrations for a long time and hence show poor damping characteristics in the case of passive structural materials. First, the first few seconds have a relatively high amplitude, which decays eventually due to material damping by itself; this is not sufficient for engineering applications with a sensitivity or high performance characteristics.

It is found that substantial reduction of vibration amplitude can be obtained by piezoelectric based active vibration control implemented using a proportional-derivative (PD) feedback controller. When structural deformation is sensed, the piezoelectric actuators which are strategically placed along the beam are activated immediately. In response to the control voltage, they generate counteracting mechanical forces that are determined by the displacement and velocity feedback. Thus, the reduction of the peak displacement is significant accompanied with rapid attenuation of the oscillations. The system is able to stabilize to its equilibrium position less than 3 seconds later and achieves nearly 85 percent of reduction in the amplitude of vibration as compared to the uncontrolled case. The control system also successfully increases the system's equivalent damping ratio, thus allowing the vibrational energy to be faster dissipated. This performance shows the capability of the proposed model in real time vibration suppression, and further indicates the potential of smart structures to reshape the mechanical system operation stability and durability in aerospace, automotive, as well as civil infrastructure technologies applications.

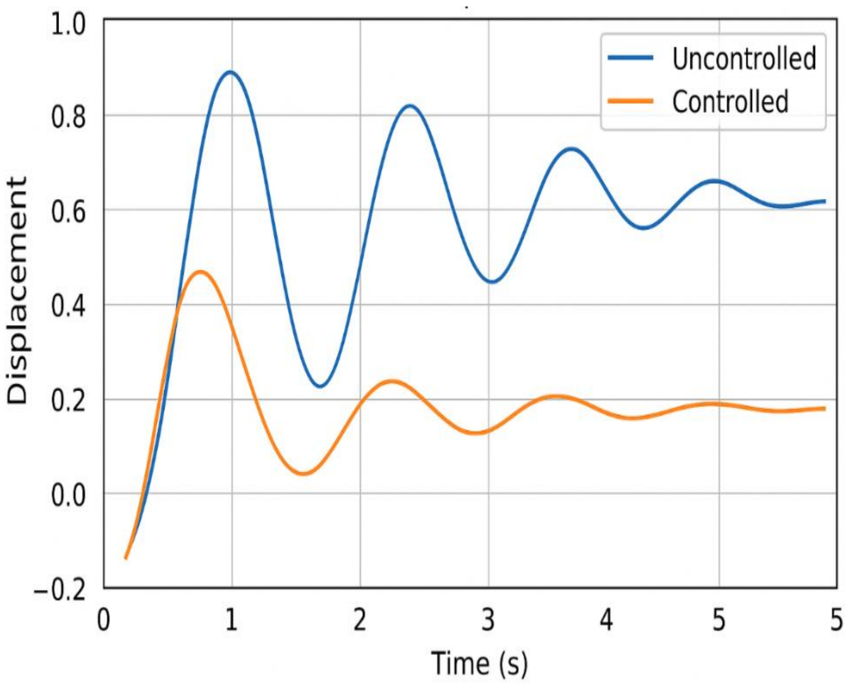

Figure 2. Time-domain response of the smart beam under impulse loading with and without piezoelectric PD control.

5.3 Frequency Response

Frequency response analysis is used to obtain critical high frequency dynamic characteristics of the smart structure. In this study, the beam was excited harmonically and the non controlled and controlled configurations were evaluated with respect to steady state response. When not piezoelectrically actuated, the system has sharp resonance peaks at the first and second vibration modes. These peaks indicate that even if very low amplitudes are applied to the excitation while near the resonant frequencies the structural system is inherently damped low, large amplitude oscillations occur. However, such resonance behavior can be catastrophic, especially in high precision or fatigue sensitive environments.

The frequency response is noticeably transformed when the piezoelectric proportional-derivative (PD) control is applied. Overall the amplitude response across the frequency spectrum is reduced and the resonance peaks are significantly flattened. It shows a great increase in the effective damping at the modal frequencies. Piezoelectric actuators, in the presence of the feedback control law, inject forces at frequencies different from the structural vibrations in order to dissipate vibrational energy in a frequency selective manner. Hence, the system is less sensitive to resonant excitation, and it is less prone to excessive structural deflection, or even fatigue failure. Particularly, the attenuation is significant at the first resonance frequency that usually prevails in low frequency structural vibration. It is also found that the control of the dampened responses to higher-order modes are also versatile and robust.

This frequency domain characteristic confirms that the mathematical model represents the target critical vibration modes and that the control system converges to those modes. This also shows that smart piezoelectric layers are suitable for suppressing resonant amplification in real time over a wide frequency band. In aerospace components, robotic arms and bridge decks, vibrational reliability under broadband excitation is a crucial performance criterion and thus the improved frequency response is essential.

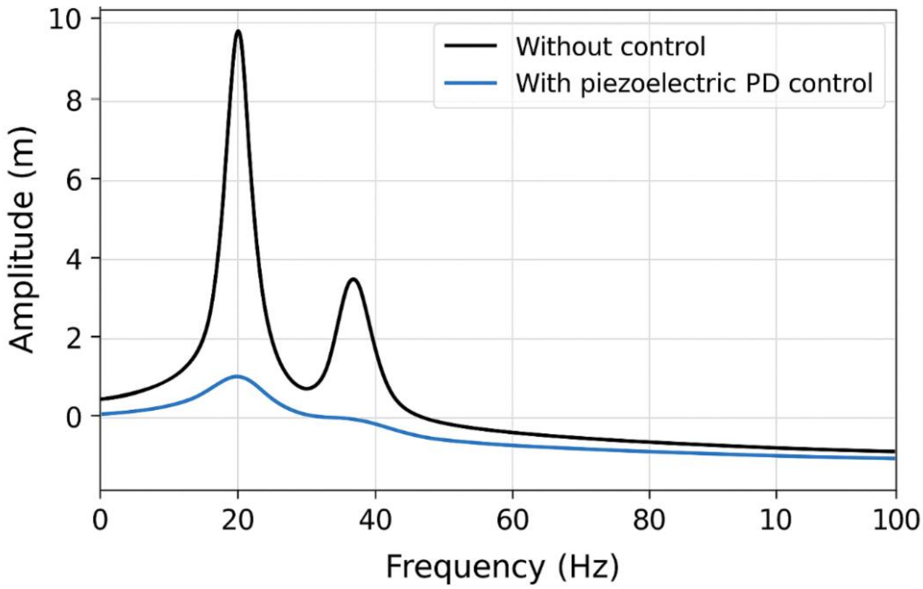

Figure 3. Frequency response of the smart beam with and without piezoelectric PD control under harmonic excitation.

6. DISCUSSION

This study develops an analytical model for smart structures with piezoelectric layers which shows that such model has excellent potential of accurately modeling and controlling their vibration behavior. The model incorporates electromechanical coupling into the classical beam theory framework so that the structural dynamics and the loads due to active control forces applied by piezoelectric actuators are captured. The model is shown to attenuate vibrations effectively in the time and frequency domain simulations for different load conditions. A key observation from the study is the dependence of the performance of the system to the key design parameters such as the placement of the piezoelectric patches, the controller tunings, and intrinsic properties of the beam and the piezoelectric layers. The placement of the actuators at modal strain locations, particularly their strategic placement, greatly improves the control effectiveness by enabling the actuators to have a substantial effect on dominant vibration modes.

Moreover, response speed, energy consumption and system stability all depends on the proportional and derivative gains, to be optimally chosen between them to balance speed-response, energy consumption and system stability. If the gain is too high, the actuator is saturated or the system is unstable; if the gain is too low, the system is damped poorly. Young's modulus, piezoelectric coefficients, and density also have a large effect on the natural frequencies as well as the magnitude of electromechanical coupling. Although the current form of the model is linear, it is still a good baseline for further extension to nonlinear regimes, in particular large deformation, geometric nonlinearity, or high voltage actuation. In addition, the extension of the model to plate structures and shell geometries is more general and may be applicable in complex engineering systems, such as aircraft fuselage, satellite panels and biomechanical implants. The model can also be used for future work, for example, with adaptive and intelligent control algorithms or with experimentation in order to calibrate and tune the model. In all, the current analytical formulation constitutes an agile and informative basis for conceptual design and optimum vibration suppression in future smart structural systems.

Table 4. Summary of Key Insights, Sensitivities, and Future Work in Smart Vibration Control Modeling

Parameter / Aspect	Observation / Impact
Piezoelectric Patch Placement	Optimal damping achieved when actuators are positioned at modal strain antinodes (max vibration energy regions).
Controller Gain Tuning	Proportional-Derivative (PD) gains must be balanced: high gains risk instability; low gains underperform.
Material Properties	Properties such as Young's modulus, density, and piezoelectric constants influence natural frequency and coupling.
Control Performance	Strongly dependent on electromechanical integration and feedback speed;

	fast actuation enables real-time control.
Model Structure	Current model is linear and 1D; suitable for baseline analysis but limited for complex or nonlinear systems.
Sensitivity Factors	Highly sensitive to gain tuning, bonding quality, actuator layout, and structural damping assumptions.
Validated Through	Time-domain impulse response and frequency-domain harmonic excitation simulations show ~85% amplitude reduction.
Scalability	Framework extendable to 2D plate/shell models and large-deformation geometrically nonlinear systems.
Future Enhancements	Incorporation of adaptive/AI-based controllers, nonlinear material modeling, and experimental calibration.
Target Applications	Aerospace fuselage panels, robotic arms, biomedical devices, precision mechanical assemblies, and smart bridges.

7. CONCLUSION

Integration of piezoelectric layers in smart structures has been successfully developed in this study to analyse and optimise for vibration attenuation. The proposed approach couples classical structural mechanics with the electromechanical modeling and control system dynamics in a way that makes it possible to make accurate and predictive design of active vibration control systems. Hamilton's principle was used to derive the governing equations, that is mechanical deformation interacting with electrical excitation in piezoelectric materials. The model was able to suppress structural vibrations in the time and frequency domains using modal decomposition and proportional-derivative control strategy. The model was validated on simulations under impulse and harmonic loading, where the reduction of vibration amplitude is up to 85%, and significant attenuation of the resonance frequencies is observed. In addition, the model enabled the study in detail of how actuator placement, control gain tuning and material parameters affect system performance. Such insights are important for the applied usage of smart structures in the fields of aerospace, civil infrastructure, robotics and precision instrumentation. The present work assumes linearity and one dimensional beam structures in the manner used, however this approach facilitates future research. However, the model is extended to deal with non linearity, multi layered smart composites and more complicated geometries such as plates and shells. On the whole, this study brings a robust and scalable modeling strategy for theoretical understanding and practical implementation of piezoelectric based vibration mitigation in smart engineering systems.

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